Mean field super-Brownian motion

Yaozhong Hu

University of Alberta at Edmonton

jointly with
Michael A. Kouritzin (University of Alberta)
Panqiu Xia (Auburn University)
Jiayu Zheng (MSU-BIT University)

18th Workshop on Markov Processes and Rel. Topics
BNU and Tianjin University

07/30 - 08/02/2023

Based on the work

(1) Y. Hu ; M. A. Kouritzin; Panqiu Xia; Jiayu Zheng

On mean-field super-Brownian motions.

Ann. Applied Probab. to appear

(2) Y. Hu; Xiong Wang; Panqiu Xia; Jiayu Zheng

Moment asymptotics for super-Brownian motions.

Submitted.

Outline

1. Branching mechanism of particles
2. Mean field super-Brownian motion
3. Moment formula
4. Moment bound for super Brownian motion
5. Limit of branching particles

1. Branching mechanism of particles

$n \in \mathbb{N}$ is a rescaling parameter and δ is a smoothing parameter.

1. Start at $t = 0$ with K_n branching particles (first generation particles) spatially distributed in $\mathbb{R}$ at points $x_1, \cdots, x_{K_n}$. The initial measure is defined as

$$X_0^{\delta,n} = X_0^{\delta,n} = \frac{1}{n} \sum_{i=1}^{K_n} \delta_{x_i} \,.$$

2. Each particle independently moves according to a Brownian motion with an independent lifespan of exponential time with some parameter depending on the environment. At the time of its death the particle produces 0 or 2 off-springs with equal probability.

To describe in more details the branching mechanism we use the collection of all multi-indices

$$\mathcal{I} = \{\alpha = (\alpha_0, \ldots, \alpha_N) : N \in \mathbb{N}, \alpha_0 \in \mathbb{N}, \alpha_i \in \{1, 2\}, 1 \leq i \leq N\}$$

to label all possible particles in the system. Thus by definition of $\mathcal{I}$, we see that each particle is allowed to generate at most 2 offspring. For example, $\alpha = (3, 1)$ means the elder successor of the third particle of the first generation. The particle $\alpha = (3, 1)$ does not produce the third generation.

For any $\alpha = (\alpha_0, \ldots, \alpha_N)$, we write $\alpha - 1 = (\alpha_0, \ldots, \alpha_{N-1})$. Then, $\alpha - 1$ is uniquely determined as the mother of particle α and we can define $\alpha - 2$, $\alpha - 3$, ... and $\alpha - N = (\alpha_0)$ iteratively.

The initial position of each particle inherits her mother's death position, and its motion can be described by B^α before she dies. To be more precise, denote by $\beta^{\delta,n}(\alpha)$ and $\zeta^{\delta,n}(\alpha)$ the birth and death times of the particle $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_N)$. The notation

$$\alpha \sim_n t \qquad \Longleftrightarrow \qquad \beta^{\delta,n}(\alpha) \le t < \zeta^{\delta,n}(\alpha)$$

means that the particle is still alive at time t. Let $\{B_t^\alpha, \alpha \in \mathcal{I}\}$ be a family of independent Brownian motions. During its lifetime, the particle α moves according to

$$\xi_t^\alpha = \xi^{\alpha-1}_{\beta^{\delta,n}(\alpha)} + B_t^\alpha - B^\alpha_{\beta^{\delta,n}(\alpha)}, \quad \beta^{\delta,n}(\alpha) \le t < \zeta^{\delta,n}(\alpha),$$

which is defined recursively.

The notation

$$X_t^{\delta,n} = \frac{1}{n} \sum_{\alpha \sim_n t} \delta_{\xi_t^\alpha},$$

the empirical measure of the system where the summation over $\alpha \sim_n t$ is among all particles "alive" at time t (to be defined later). We also associate a smoothing random field $Y^{\delta,n}$ on $\mathbb{R}_+ \times \mathbb{R}$ given by

$$Y_t^{\delta,n}(x) = \langle X_t^{\delta,n}, p_\delta(x - \cdot)\rangle = \int_{\mathbb{R}} p_\delta(x-y) X_t^{\delta,n}(dy).$$

The lifetime of each particle α is controlled by an independent exponential clock. The parameter of each clock is $n\widetilde{\sigma}_\delta^2(t, \xi_t^\alpha, \mathbb{P}_{Y_t^{\delta,n}})$, where $\widetilde{\sigma}_\delta$ is defined by

$$\widetilde{\sigma}_\delta(t, x, \Gamma) = \int_{\mathbb{R}} dy p_\delta(x-y)\sigma\big(t, y, \Gamma(y)\big) \tag{1}$$

with some measurable function $\sigma : \mathbb{R}_+ \times \mathbb{R} \times \mathscr{P}(\mathbb{R}_+) \to \mathbb{R}_+$, $\Gamma : \mathbb{R} \to \mathscr{P}(\mathbb{R}_+)$. This means for any living particle α at time $t \geq 0$ with position ξ_t^α, the probability that she dies in the time interval $[t, t+\Delta t)$ is

$$n\widetilde{\sigma}_\delta^2\big(t, \xi_t^\alpha, \mathbb{P}_{Y_t^{\delta,n}}\big)\Delta t + o(\Delta t).$$

2. Mean field superprocess

With the above branching mechanism when $n \to \infty$ and when $\delta \to 0$, the process

$$X_t^{\delta,n} = \frac{1}{n} \sum_{\alpha \sim_n t} \delta_{\xi_t^\alpha},$$

would converge to the following SPDE

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sigma\left(t, x, \mathbb{P}_{X_t(x)}\right) \sqrt{X_t(x)} \dot{W}(t, x), \qquad (2)$$

where $\mathbb{P}_{X_t(x)}$ is the probability law of the real valued random variable $X_t(x)$.

We shall focus on the existence, uniqueness and regularity of the solution to equation (2). The first difficulty that we encounter is that there exists no readily-applicable, fully-developed theory on the Fokker-Planck-Kolmogorov equation associated with (2). So, we cannot follow the approach used in finite dimensional case to study the existence and uniqueness of solutions to the associated Fokker-Planck-Kolmogorov equation first, and then to solve the mean field equation.

Buckdahn, R., Li, J., Peng, S., and Rainer, C.

Mean-field stochastic differential equations and associated PDEs.

Ann. Probab. 45, **2** (2017), 824–878.

Hu, Y.; Kouritzin, M.; Zheng, Jiayu

Nonlinear McKean-Vlasov diffusions under the weak Hörmander condition with quantile-dependent coefficients

To appear in Potential Analysis

Nevertheless, the anticipation of solutions to (2) is well justified. Due to the appearance of the branching character (the $\sqrt{X_t(x)}$ factor in front of the noise), it is natural to use a branching particle system to approximate this equation. Assuming that such approximation is done and some high-density limit exists, one presumably obtains that every limit point $X = X_t(dx, \omega)$ is an $\mathcal{M}_F(\mathbb{R})$-valued Markov process.

Let $\mathcal{M}_F(\mathbb{R})$ be the set of all finite measures on $\mathbb{R}$, let $\mathscr{P}(\mathbb{R}_+)$ be the collection of all Borel probability measures on $\mathbb{R}_+$ equipped with the weak topology, namely, $\lim_{n\to\infty} \mathbb{P}_n = \mathbb{P}$ in $\mathscr{P}(\mathbb{R}_+)$, denoted by $\mathbb{P}_n \Rightarrow \mathbb{P}$, if

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \phi(x)\mathbb{P}_n(dx) = \int_{\mathbb{R}_+} \phi(x)\mathbb{P}(dx),$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. We write $\mathcal{M}(\mathbb{R}; \mathscr{P}(\mathbb{R}_+))$ for the collections of measurable functions on $\mathbb{R}$ with values in $\mathscr{P}(\mathbb{R}_+)$.

Hypothesis (Hypothesis 1)

(i) *σ^2 is positive and bounded, that is, there exists a positive constant K_0 such that*

$$0 < \sigma^2(t, x, \mu) \leq K_0$$

for all $(t, x, \mu) \in \mathbb{R}_+ \times \mathbb{R} \times \mathscr{P}(\mathbb{R}_+)$.

(ii) *σ^2 is continuous with respect to all the arguments, in the sense that for any $(t_n, x_n) \to (t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\mu_n \Rightarrow \mu$ in $\mathscr{P}(\mathbb{R}_+)$, it follows that*

$$\lim_{n\to\infty} \sigma^2(t_n, x_n, \mu_n) = \sigma^2(t, x, \mu).$$

Hypothesis (Hypothesis 2)

For any $(t, x, \mu) \in [0, T] \times \mathbb{R} \times \mathscr{P}(\mathbb{R}_+)$,

$$\sigma^2(t, x, \mu) = f(t, x, \mathbb{E}[X_\mu], \mathbb{E}[X_\mu^2], \ldots, \mathbb{E}[X_\mu^N]) \,,$$

where $N \in \mathbb{N}$, X_μ *is a random variable with distribution* μ *and* f *is a continuous function on* $[0, T] \times \mathbb{R} \times \mathbb{R}_+^N$ *that is positive and bounded. Moreover,* f *is assumed to be differentiable in the last* N *spatial arguments with bounded derivatives.*

For any $\gamma \in \mathbb{R}$, the Hilbert space $\mathcal{H}_\gamma$ is a collection of real sequences, namely, $x = (x_i)_{i\in\mathbb{N}}$ with $x_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, equipped with inner product

$$\langle x, y\rangle_{\mathcal{H}_\gamma} = \sum_{n=1}^{\infty} (n!)^{-2\gamma} x_n y_n, \tag{3}$$

for all $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$.

Hypothesis (Hypothesis 3)

Let $\mathcal{H} = \mathcal{H}_\gamma$ with $\gamma > \frac{3}{2}$, and let $\mathcal{H}_+ = \{x = (x_i)_{i\in\mathbb{N}} \in \mathcal{H} : x_i \geq 0, \forall i \geq 1\}$. Then, σ can be represented as $\sigma(t, x, \mu)^2 = f(t, x, \mathbb{E}[X_\mu], \mathbb{E}[X_\mu^2], \dots)$ for some measurable function f on $[0, T] \times \mathbb{R} \times \mathcal{H}_+$ that is positive and bounded. Moreover, f is Lipschitz in $y \in \mathcal{H}_+$ with uniform constant in $(t, x) \in [0, T] \times \mathbb{R}$, namely,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |f(t, x, y_1) - f(t, x, y_2)| \leq L\|y_1 - y_2\|_{\mathcal{H}},$$

for all $y_1, y_2 \in \mathcal{H}_+$ with some constant $L > 0$.

Next, we state the last hypothesis about the initial condition X_0.

Hypothesis (Hypothesis 4)

$X_0 \in \mathcal{M}_F(\mathbb{R})$ has a bounded density, still denoted by X_0, such that $X_0 \in H_{1,2}(\mathbb{R})$, namely, $\|X_0\|_{1,2} = \|X_0\|_2 + \|\nabla X_0\|_2 < \infty$.

Now, we are ready to state the main results

Theorem (Hu-Kouritzin-Xia-Zheng)

Assume X_0 *satisfying Hypothesis 4. Then, equation* (2) *with initial condition* X_0 *has a weak solution on any time interval* $[0, T]$ *under one of Hypotheses 1, 2 and 3. Additionally, the solution is unique in distribution under either Hypothesis 2 or 3.*

3. Moment formula

$$X_t(x) = \int_{\mathbb{R}} dy p_t(x-y) X_0(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(s, y, \mathbb{P}_{X_s(y)})\sqrt{X_s(y)}W(ds, dy).$$

Due to the singularity of $p_{t-s}(x-y)$ when $t = s$, $X_t(x)$ is not a semimartingale in t. We introduce an auxiliary process $Y^t = \{Y_s^t(x) : 0 \leq s \leq t, x \in \mathbb{R}\}$, where

$$Y_s^t(x) = \int_{\mathbb{R}} dy p_t(x-y) X_0(y) + \int_0^s \int_{\mathbb{R}} p_{t-r}(x-y)\sigma(r, y, \mathbb{P}_{X_r(y)})\sqrt{X_r(y)}W(dr, dy). \quad (4)$$

Theorem (Hu-Kouritzin-Xia-Zheng)

Suppose that $X_0 \in \mathcal{M}_F(\mathbb{R})$ *satisfies Hypothesis 4. Let* $n \in \mathbb{N}$*. Then, for any* $(t,x) \in [0,T] \times \mathbb{R}$*, the following equation holds:*

$$\begin{aligned}
\mathbb{E}[X_t(x)^n] = \sum_{n'=0}^{n-1} \sum_{(\alpha,\beta,\tau)\in\mathcal{J}_{n,n'}} \prod_{i=1}^{n} \Big(\int_{\mathbb{R}} dz p_t(x-z) X_0(z) \Big)^{1-\alpha_i} \\
\times \int_{\mathbb{T}^t_{n'}} d\mathbf{s}_{n'} \int_{\mathbb{R}^{n'}} d\mathbf{z}_{n'} \prod_{i=1}^{n'} \Big(\int_{\mathbb{R}} dz p_{s_i}(z_i - z) X_0(z) \Big)^{1-\beta_i} \\
\prod_{i=1}^{|\alpha|} p(t - s_{\tau(i)}, x - z_{\tau(i)}) \\
\times \prod_{i=|\alpha|+1}^{2n'} p(s_{\iota_\beta(i-|\alpha|)} - s_{\tau(i)}, z_{\iota_\beta(i-|\alpha|)} - z_{\tau(i)}) \\
\times \prod_{i=1}^{n'} \sigma(s_i, z_i, \mathbb{P}_{X_{s_i}(z_i)})^2,
\end{aligned} \tag{5}$$

Theorem (continued)

where the set $\mathcal{J}_{n,n'}$ *of triples* (α, β, τ) *are some index set which is complicated to describe here.*

$$\mathbb{T}_{n'}^{t} = \left\{ \mathbf{s}_{n'} = (s_1, \ldots, s_{n'}) \in [0, T]^{n'} : 0 < s_{n'} < s_{n'-1} < \cdots < s_1 < t \right\}, \tag{6}$$

and $p(t, x) = p_t(x)$ *to avoid long sub-indexes.*

Proposition

Assume that $X_0 \in \mathcal{M}_F(\mathbb{R})$ *satisfies Hypothesis 4 and let* $X = \{X_t(x) : (t,x) \in [0,T] \times \mathbb{R}\}$ *be a solution to equation* (2). *Then,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}[X_t(x)^n] \le c_1 c_2^n (n!)^{\frac{3}{2}}, \tag{7}$$

with constants $c_1, c_2 > 0$ *independent of n.*

Use this bound we can prove the weak uniqueness for equation (2).

4. Moment bounds for super Brownian motion

Return to the classical super Brownian motion:

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sqrt{u_t(x)}\,\dot{W}(t,x), \tag{8}$$

where $\dot{W}$ denotes the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$.

Hypothesis (Hypothesis 5)

u_0 is a positive function on $\mathbb{R}$ that is two-sided bounded by positive constants, namely,

$$K_1 \leq u_0(x) \leq K_2,$$

for all $x \in \mathbb{R}$ with $K_2 \geq K_1 > 0$.

Hypothesis (Hypothesis 6)

u_0 is a finite measure on $\mathbb{R}$ such that for any $x \in \mathbb{R}$,

$$\lim_{t\uparrow\infty} t^{\gamma} \int_{\mathbb{R}} p_t(x-z)u_0(dz) = L \in (0,\infty), \tag{9}$$

with some $\gamma \in (0,1)$, where $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ denotes the heat kernel.

Theorem (Hu-Wang-Xia-Zheng)

Let $u = \{u_t(x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to (8)*, and let n be any positive integer. Then, under Hypothesis 5,*

$$K_*^n(1 + n!t^{\frac{1}{2}(n-1)}) \le \mathbb{E}(u_t(x)^n) \le (K^*)^n(1 + n!t^{\frac{1}{2}(n-1)}), \qquad (10)$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Instead, under Hypothesis 6,

$$K_*^n n!t^{\frac{n-1}{2}-\gamma} \le \mathbb{E}(u_t(x)^n) \le (K^*)^n n!t^{\frac{n-1}{2}-\gamma}, \qquad (11)$$

for all $(t,x) \in [nC_x \vee 1, \infty) \times \mathbb{R}$ where $C_x > 0$, depending on x, is such that

$$\frac{1}{2}L \le t^\gamma \int_{\mathbb{R}} p_t(x-z)u_0(dz) \le 2L$$

for all $t > C_x$. Especially, the second inequality in (11) *holds for all $(t,x) \in [C_x, \infty) \times \mathbb{R}$. Here, K_* and K^* are positive constants independent of n, t and x.*

5. Limit of branching particles

(i) $(X^\delta, Y^\delta) = (X^\delta, \langle X^\delta, p_\delta(x-\cdot)\rangle)$ is a solution to following martingale problem:

$$M_t(\phi) = \langle X_t, \phi\rangle - \langle X_0, \phi\rangle - \frac{1}{2}\int_0^t \langle X_s, \phi\rangle ds \tag{12}$$

is a square integrable martingale with quadratic variation

$$\langle M(\phi)\rangle_t = \int_0^t \int_{\mathbb{R}} \sigma(s, x, \mathbb{P}_{X_s(x)})^2 \phi(x)^2 X_s(dx) ds. \tag{13}$$

(2) If $(X^\delta, Y^\delta) = (X^\delta, \langle X^\delta, p_\delta(x-\cdot)\rangle)$ is a solution to martingale problem MP (12) and (13) with initial condition $X_0 \in \mathcal{M}_F(\mathbb{R})$. Then, for every $t \in \mathbb{R}_+$, X_t^δ has a Lebesgue density. Moreover, identifying $X_t^\delta(x)$ as the density of X_t^δ, the pair (X^δ, Y^δ) satisfies equation (14) for some space-time white noise W.

$$\begin{cases} \frac{\partial}{\partial t} X_t^\delta(x) = \frac{1}{2}\Delta X_t^\delta(x) + \widetilde{\sigma}_\delta(t, x, \mathbb{P}_{Y_t^\delta})\sqrt{X_t^\delta(x)}\dot{W}(t,x), \\ Y_t^\delta(x) = \int_{\mathbb{R}} p_\delta(x-y) X_t^\delta(dy), \end{cases} \tag{14}$$

tightness criteria

A family $\mathcal{P}$ of probability measures on $C([0,T]\times\mathbb{R};\mathbb{R})$ is precompact if

(i) $\lim_{A\uparrow\infty}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}(|X_t(0)|>A)=0.$

(ii) For each $x\in\mathbb{R}$ and $\rho>0$,

$$\lim_{\epsilon\downarrow 0}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\Big(\sup_{0\le s\le t\le T,|t-s|<\epsilon}|X_t(x)-X_s(x)|>\rho\Big)=0.$$

(iii) For every $R>0$ and $\rho>0$,

$$\lim_{\epsilon\downarrow 0}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\Big(\sup_{0\le t\le T,-R\le x\le y\le R,|x-y|<\epsilon}|X_t(x)-X_t(y)|>\rho\Big)=0.$$

Proposition

Assume Hypothesis. Then, $\{Y^\delta\}_{\delta>0}$ *is a tight sequence in* $C([0,T]\times\mathbb{R};\mathbb{R})$.

We only sketch the proof of the theorem under Hypothesis 1. For other situations under Hypothesis 2 or 3, we can modify the proof following the idea. Let (X^δ, Y^δ) be a solution to (14). Then, there exists a sequence $\delta_n \downarrow$ such that Y^{δ_n} is convergent in distribution in $C([0,T] \times \mathbb{R}; \mathbb{R})$ to some random field Y. By the typical tightness argument, one can show that $\{X^{\delta_n} : n \geq 1\}$ is tight in $D([0,T]; \mathcal{M}_F(\mathbb{R}))$. Therefore, by taking subsequence of $\{X^{\delta_n}\}$, we can assume it converges in distribution to some $\mathcal{M}_F(\mathbb{R})$-valued process X. By the Skorohod representation theorem, we can also assume this convergence is almost surely.

In the next step, we show that (X, Y) is a weak solution to the following equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2} \Delta X_t(x) + \sigma(t, x, \mathbb{P}_{Y_t(x)}) \sqrt{X_t(x)}\, \dot{W}(t, x).$$

Equivalently, it suffices to show that X_t is a solution to the following martingale problem), for any $\phi \in \mathcal{S}(\mathbb{R})$,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \frac{1}{2}\int_0^t X_s(\Delta\phi)ds \tag{15}$$

is a continuous square integrable martingale, with quadratic variation

$$\langle M(\phi)\rangle_t = \int_0^t \sigma(s, x, \mathbb{P}_{Y_s(x)})^2 \phi(x)^2 X_s(dx)ds. \tag{16}$$

Notice that, using a elementary theorem, we have

$$\begin{aligned}\langle M(\phi)\rangle_t &= \lim_{n\to\infty} \langle M^{\delta_n}(\phi)\rangle_t \\ &= \lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}} X_s^{\delta_n}(dx)\Big(\int_{\mathbb{R}} dy p_{\delta_n}(x-y)\sigma(s, y, \mathbb{P}_{Y_s^{\delta_n}(y)})\Big)^2 \phi(x)^2.\end{aligned}$$

To verify the limit, we compute the following quantity,

$$\Big|\int_0^t ds\int_{\mathbb{R}} X_s^{\delta_n}(dx)\Big(\int_{\mathbb{R}} dy p_{\delta_n}(x-y)\sigma(s,y,\mathbb{P}_{Y_s^{\delta_n}(y)})\Big)^2\phi(x)^2 - \int_0^t ds\int_{\mathbb{R}}\sigma(s,x,\mathbb{P}_{Y_s(x)})^2\phi(x)^2 X_s(dx)\Big| \le I_1 + I_2$$

where

$$I_1 = \Big|\int_0^t ds\int_{\mathbb{R}}\Big[\Big(\int_{\mathbb{R}} dy p_{\delta_n}(x-y)\sigma(s,y,\mathbb{P}_{Y_s^{\delta_n}(y)})\Big)^2 - \sigma(s,x,\mathbb{P}_{Y_s(x)})^2\Big]\phi(x)^2 X_s^{\delta_n}(dx)\Big|$$

and

$$I_2 = \Big|\int_0^t ds\int_{\mathbb{R}} dx\sigma(s,x,\mathbb{P}_{Y_s(x)})^2\phi(x)^2\big(X_s^{\delta_n}(dx) - X_s(dx)\big)\Big|.$$

It is clear that $I_2 \to 0$ as $n \to \infty$ because $X^{\delta_n} \to X$ in $D([0,T];\mathcal{M}_F(\mathbb{R}))$. On the other hand, notice that $X_s^{\delta_n}$ has a density almost surely. Thus, by Cauchy-Schwarz's inequality

$$I_1 \leq \left(\int_{\mathbb{R}} dx \left[\left(\int_{\mathbb{R}} dy p_{\delta_n}(x-y) \sigma(s, y, \mathbb{P}_{Y_s^{\delta_n}(y)}) \right)^2 - \sigma(s, x, \mathbb{P}_{Y_s(x)})^2 \right]^2 \phi(x)^2 \right)^{\frac{1}{2}}$$

$$\times \int_0^t \left(\int_{\mathbb{R}} \phi(x)^2 X_s^{\delta_n}(x)^2 dx \right)^{\frac{1}{2}} := I_{11} \times I_{12}.$$

By dominated convergence theorem, we know that $I_{11} \to 0$ as $n \to \infty$. Furthermore, one can show that $\mathbb{E}[I_{12}]$ is uniformly bounded in n. As a consequence, it follows by Fatou's lemma that

$$\mathbb{E}\Big[\liminf_{n\to\infty} I_1\Big] \leq \lim_{n\to\infty} \mathbb{E}[I_1] = 0.$$

This implies that $\liminf_{n\to\infty} I_1 = 0$ almost surely. That is enough to prove (16) because we can take subsequence so that the above $\liminf_{n\to\infty}$ can be replaced by $\lim_{n\to\infty}$.

Finally, we complete the proof of this theorem by showing that for any $t \in [0, T]$, the distribution of X_t and Y_t coincide. Indeed, for any $\phi \in \mathcal{S}(\mathbb{R})$, we can show that

$$\begin{aligned}\mathbb{E}[\langle X_t, \phi\rangle] - \mathbb{E}[\langle Y_t, \phi\rangle] \leq & \left|\mathbb{E}[\langle X_t, \phi\rangle] - \mathbb{E}[\langle X_t^{\delta_n}, \phi\rangle]\right| + \left|\mathbb{E}[\langle Y_t, \phi\rangle] - \mathbb{E}[\langle Y_t^{\delta_n}, \phi\rangle]\right| \\ & + \left|\mathbb{E}[\langle X_t^{\delta_n}, \phi\rangle] - \mathbb{E}[\langle Y_t^{\delta_n}, \phi\rangle]\right|.\end{aligned}$$

It suffices to show the convergence to 0 of the last term. Recall that $Y_t^{\delta_n}(x) = \langle X_t^{\delta_n}, p_\delta(x-\cdot)\rangle$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Thus, we can write

$$\begin{aligned}\left|\mathbb{E}[\langle X_t^{\delta_n}, \phi\rangle] - \mathbb{E}[\langle Y_t^{\delta_n}, \phi\rangle]\right| =& \mathbb{E}\left[\left| \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy p_{\delta_n}(x-y)(\phi(x)-\phi(y)) X_t(dx)\right|\right] \\ \leq& \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} dy p_{\delta_n}(x-y)(\phi(x)-\phi(y))\right| \mathbb{E}[\langle X_t, \mathbf{1}\rangle].\end{aligned}$$

Notice that

$$\begin{aligned}&\left| \int_{\mathbb{R}} dy p_{\delta_n}(x-y)(\phi(x)-\phi(y))\right| \\ \leq& \int_{|x-y|\leq \delta_n^{\frac{1}{3}}} dy p_{\delta_n}(x-y)|\phi(x)-\phi(y)| + 2\|\phi\|_\infty \int_{|x-y|>\delta_n^{\frac{1}{3}}} dy p_{\delta_n}(x-y) \\ \leq& \|\phi\|_{1,\infty} \delta_n^{\frac{1}{3}} \int_{|z|\leq \delta_n^{\frac{1}{3}}} dz p_{\delta_n}(z) + 2\|\phi\|_\infty \int_{|z|>\delta_n^{-\frac{1}{6}}} dz \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \to 0,\end{aligned}$$

as $n \to \infty$. As a consequence, we have $\mathbb{E}[\langle X_t, \phi\rangle] = \mathbb{E}[\langle Y_t, \phi\rangle]$ for all $\phi \in \mathcal{S}(\mathbb{R})$. The proof of the existence part of the theorem is complete.

THANKS